A Proofs

A.1 Proof of Proposition 1

Proof. The first step is to solve for the closed form solution of the optimal value function. I show in Lemma 1 that equation (7) is solvable and the solution contains a free parameter which is pinned down by the liquidity constraint.

Lemma 1. *The analytical solution.*
Suppose that there exists a function $c(w)$ such that

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - K_0 \left(ac(w) + b\right)^{-\frac{r}{a(\rho - r)}} \tag{1}$$

where K_0 is a constant. Then equation (7) holds for the HARA utility function (2).

Proof. Suppose that (1) holds. Differentiating both sides with respect to w, I get

$$c'(w) = r + a(\rho - r) + \frac{r}{a(\rho - r)} K_0 \left(ac(w) + b\right)^{-\frac{r + a(\rho - r)}{a(\rho - r)}} ac'(w). \tag{2}$$

Solving (1) for the K_0-term

$$K_0 \left(ac(w) + b\right)^{-\frac{r}{a(\rho - r)}} = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - c(w). \tag{3}$$

Substituting (3) into (2)

$$c'(w) = r + a(\rho - r) + \frac{r}{\rho - r}((rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - c(w))\left(ac(w) + b\right)^{-1} c'(w). \tag{4}$$

Rearranging (4)

$$c(w) = \frac{r + a(\rho - r)}{r}\left(\frac{ac(w) + b}{c'(w)}(\rho - r) + rw + y\right) - (ac(w) + b)\frac{\rho - r}{r} + b\frac{\rho - r}{r}$$

$$c(w) = rw + y + \frac{ac(w) + b}{c'(w)}(\rho - r)$$

which is (7). □

The second step is to show that there exists a $c(w)$ that satisfies (1) together with the liquidity constraint $c(\underline{w}) \leq r\underline{w} + y$. Note that under the assumption of Proposition 2, $\rho > r > 0$ so $c(w) \geq rw + y$. Consumption must therefore be equal to income at the

constraint, $c(\underline{w}) = r\underline{w} + y$. Solve (1) for K_0 using the liquidity constraint

$$K_0 = \frac{\rho - r}{r}\Big(a(r\underline{w} + y) + b\Big)^{\frac{r+a(\rho-r)}{a(\rho-r)}}. \tag{5}$$

Insert this K_0 into (1)

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\Big(a(r\underline{w} + y) + b\Big)\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} \tag{6}$$

By the assumption in Proposition 2, $\rho > r > 0$ and $ac(w) + b > 0$, so (6) is continuous and exists for all $w \in [\underline{w}, \infty)$. Furthermore, it satisfies the liquidity constraint by construction.

The third step is to derive the solution for optimal consumption in the unconstrained case. This is very helpful because consumption in the constrained case is always bounded by the linear unconstrained solution and I will use this property to show that (6) is optimal. I define unconstrained consumption as the solution to (6) under the natural borrowing limit ($\underline{w} = -\frac{y}{r}$)

$$c_u(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r}. \tag{7}$$

Unconstrained consumption is thus linear in total income. Moreover,

$$c_u(w) - c(w) = \frac{\rho - r}{r}\Big(a(r\underline{w} + y) + b\Big)\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} > 0$$

so constrained consumption is always bounded above by the linear unconstrained consumption function.

The fourth step is to verify that the consumption rule (6) is indeed optimal. Since the first-order condition and the liquidity constraint hold by construction, it suffices to verify the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} V(w_t) = 0$$

First note that w is always bounded below by the liquidity constraint. I therefore only need to show that w is bounded above. Note first that consumption is always greater or equal to total income

$$c(w) - (rw + y) = (a(rw + y) + b)\frac{\rho - r}{r} - \Big(a(r\underline{w} + y) + b\Big)\frac{\rho - r}{r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} \geq 0$$

since $w \geq \underline{w}$ and $0 \leq \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} \leq 1$ when $\rho > r > 0$. It follows that $w_{t+\Delta t} \leq w_t$ for all t such that the path of wealth is also bounded ($w_t \in [\underline{w}, w_0]$ for all t). Further, the

value function is continuous since the utility function is continuous, implying that the transversality condition holds when w is bounded above.

□

A.2 Proof of Proposition 2

Proof. Proposition 2 follows by applying repeated total derivation on the solution for optimal consumption in the constrained problem and the unconstrained problem, and comparing the results. For convenience, I restate the optimal consumption functions in the constrained

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}(ac(w) + b)^{-\frac{r}{a(\rho-r)}}$$

and unconstrained case

$$c_u(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r}.$$

Part I: $c_u(w) > c(w) > rw + y$
Calculating the difference between consumption in the constrained and the unconstrained case, I get

$$c(w) - c_u(w) = -\frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}(ac(w) + b)^{-\frac{r}{a(\rho-r)}} < 0$$

where the last inequality follows since $\rho > r > 0$ and $ac(w) + b > 0$ by assumption. To show that consumption is always greater than income, calculate

$$c(w) - (rw + y) = (a(rw + y) + b)\frac{\rho - r}{r} - \left(a(r\underline{w} + y) + b\right)\frac{\rho - r}{r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} > 0$$

since $w \in (\underline{w}, \infty)$ and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}} < 1$ when $\rho > r > 0$.

Part II: $c'(w) > c_u'(w) > 0$
By differentiating the unconstrained solution, I get

$$c_u'(w) = r + a(\rho - r) > 0$$

where the last inequality follows since I assume that $r + a(\rho - r) > 0$. I then apply total derivation on the constrained solution to find the constrained marginal propensity to

consume

$$c'(w) = r + a(\rho - r) + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c'(w). \tag{8}$$

Note that since $r + a(\rho - r) > 0$ and $\rho > r > 0$, I have

$$0 < \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < \left(\frac{a(r\underline{w} + y) + b}{a(r\underline{w} + y) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} = 1.$$

It then follows that

$$c'(w) = \frac{r + a(\rho - r)}{1 - \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}} > r + a(\rho - r) = c_u'(w).$$

Part III: $c''(w) < c_u''(w) = 0$

Note first that since the unconstrained consumption function is linear, $c_u''(w) = 0$. I therefore only need to show that $c''(w) < 0$ in the constrained solution. Apply total derivation on (8)

$$c''(w) = \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c''(w) - \frac{r + a(\rho - r)}{\rho - r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r+2a(\rho-r)}{a(\rho-r)}} (c'(w))^2$$

$$c''(w) = -\frac{1}{1 - \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}} \frac{r + a(\rho - r)}{\rho - r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r+2a(\rho-r)}{a(\rho-r)}} (c'(w))^2 < 0$$

where the last inequality follows since $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$, $\rho > r > 0$, $r + a(\rho - r) > 0$, and $ac(w) + b > 0$ by the assumptions in Proposition 2.

Part IV: $\lim_{w\to\infty} c(w) = c_u(w)$

I know that $c'(w) > 0$ for all w so that $\lim_{w\to\infty} c(w) = \infty$. It thus follows that as long as $a > 0$

$$\lim_{w\to\infty} -\frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}} = 0$$

such that

$$\lim_{w\to\infty} c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} = c_u(w).$$

□

A.3 Proof of Proposition 3

Proof. Before I start, let me restate the optimal consumption function

$$c(w) = (rw + y)\left(\frac{r + a(\rho - r)}{r}\right) + b\frac{\rho - r}{r} - \frac{\rho - r}{r}\left(a(r\underline{w} + y) + b\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} (ac(w) + b)^{-\frac{r}{a(\rho-r)}}.$$

Part I: $\frac{dc(w)}{d\underline{w}} < 0$
Apply total derivation of the consumption expression with respect to $\underline{w}$

$$\frac{dc(w)}{d\underline{w}} = -(r + a(\rho - r))\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} + \frac{dc(w)}{d\underline{w}}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 0$$

where the last inequality follows since $r+a(\rho-r) > 0$, $\rho > r > 0$, and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$.

Part II: $\frac{dc'(w)}{d\underline{w}} > 0$
Apply total derivation on (8) with respect to w to obtain

$$c'(w) = r + a(\rho - r) + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c'(w) \tag{9}$$

Apply total derivation on (9) with respect to $\underline{w}$

$$\begin{aligned}\frac{dc'(w)}{d\underline{w}} = {} & \frac{r + a(\rho - r)}{\rho - r}\frac{r}{ac(w) + b}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} c'(w) \\ & - \frac{r + a(\rho - r)}{\rho - r}\frac{a(r\underline{w} + y) + b}{(ac(w) + b)^2}\frac{dc(w)}{d\underline{w}}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r}{a(\rho-r)}} c'(w) \\ & + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \frac{dc'(w)}{d\underline{w}} > 0\end{aligned}$$

where the last inequality follows since $r + a(\rho - r) > 0$, $\rho > r > 0$, $ac(w) + b > 0$, $c'(w) > 0$, $\frac{dc(w)}{d\underline{w}} < 0$, and $0 < \left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} < 1$.

Part III: $\frac{dc''(w)}{d\underline{w}} < 0$
Apply total derivation on (9) with respect to w to obtain

$$c''(w) = -\frac{r + a(\rho - r)}{\rho - r}\left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} \frac{(c'(w))^2}{ac(w) + b} + \left(\frac{a(r\underline{w} + y) + b}{ac(w) + b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}} c''(w). \tag{10}$$

Apply total derivation on (10) with respect to $\underline{w}$

$$\begin{aligned}\frac{dc''(w)}{d\underline{w}} = &-\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}}\underbrace{\left(\frac{r}{ac(w)+b}-\frac{a(r\underline{w}+y)+b}{(ac(w)+b)^2}\frac{dc(w)}{d\underline{w}}\right)}_{>0}\frac{(c'(w))^2}{ac(w)+b}\\
&-\frac{r+a(\rho-r)}{\rho-r}\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}\underbrace{\left(\frac{2c'(w)\frac{dc'(w)}{d\underline{w}}}{ac(w)+b}-\frac{a(c'(w))^2}{(ac(w)+b)^2}\frac{dc(w)}{d\underline{w}}\right)}_{>0}\\
&+\frac{r+a(\rho-r)}{a(\rho-r)}\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r}{a(\rho-r)}}\underbrace{c''(w)}_{<0}\underbrace{\left(\frac{ar}{ac(w)+b}-\frac{a(r\underline{w}+y)+b}{(ac(w)+b)^2}a\frac{dc(w)}{d\underline{w}}\right)}_{>0}\\
&+\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}\frac{dc''(w)}{d\underline{w}}<0\end{aligned}$$

where the last inequality follows since $r+a(\rho-r)>0$, $\rho>r>0$, $ac(w)+b>0$, $\frac{dc(w)}{d\underline{w}}<0$, $a>0$, and $0<\left(\frac{a(r\underline{w}+y)+b}{ac(w)+b}\right)^{\frac{r+a(\rho-r)}{a(\rho-r)}}<1$.

□